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On a condition of existence of non-anticipating selections

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Abstract: The hereditary selections of multi-functions play an important role in the theory of differential games in connection with the construction of resolving quasi-strategies. The existence of a non-anticipating selection of a non-anticipating multi-function is considered. In most cases important for applications, it is known that any non-anticipating multi-function with non-empty compact values has a non-anticipating selection. Namely, the result is valid when the non-anticipation property is defined by a totally ordered family in the domain of "time" variable. In this note, we show that the condition is essential: when the family is not totally ordered, there exists a hereditary multi-function with non-empty compact values that has no non-anticipating selections.

Keywords: multi-function, non-anticipating selection, quasi-strategy.

1. INTRODUCTION

The non-anticipating selections of multi-functions play an important role in the theory of differential games in connection with the construction of idealized resolving strategies — quasi-strategies. In the early works (see Nardzewski (1964); Varaiya and Lin (1969); Roxin (1969); Elliott and Kalton (1972), etc.), quasi-strategies were defined as operators on the functional spaces of control realizations or trajectories with the property of physical feasibility or non-anticipation. On the other hand, in some game–theoretical constructions, non-anticipating multi-functions and, consequently, multi-valued quasi-strategies arise in a natural way (see Chentsov (1976)). In the same years (see Chentsov (1978)), the question of the existence of a single–valued selector of the multi-function preserving the property of non-anticipation was considered for mappings on the spaces of generalized controls, where specific properties of measures were essentially used. Recently (see Serkov and Chentsov (2018)) a rather general statement of the problem was studied: it was shown for the most important cases that every non-anticipating multi-function with non-empty and compact values has a non-anticipating selection. Namely, the result is valid when the non-anticipation property is defined by a totally ordered set family in the domain of "time" variable. In this paper studying the existence of a non-anticipative selection as an independent problem, we establish that the above condition of total ordering is essential: otherwise there are examples of non-anticipating multi-functions with non-empty compact values that have no non-anticipating selections.

2. NOTATION AND DEFINITIONS

Hereinafter, we use standard set-theoretic notation (quantifiers, relations, $\varnothing$ as the empty set); $\triangleq$ means "is equal by definition." Any set whose elements are sets is called a *family*. Let $\mathcal{P}(T)$ ($\mathcal{P}'(T)$) denote the family of all (all non-empty) subsets of an arbitrary set T. If A and B are non-empty sets, then B^A denotes the set of all mappings from A to B. If $f \in B^A$ and $C \in \mathcal{P}'(A)$, then $(f\,|\,C) \in B^C$ is, by definition, the restriction of f to the set C: $(f\,|\,C)(x) \triangleq f(x)\ \forall x \in C$. For $F \in \mathcal{P}'(B^A)$, we set $\big(F\,|\,C\big) \triangleq \{(f\,|\,C) : f \in F\}$. For any set $X \neq \varnothing$ and a partial order relation $\leq \in \mathcal{P}'(X \times X)$, we denote by $(X, \leq)$ the corresponding partially ordered set (or *poset*). A set $C \subset X$ is called a *chain* in $(X, \leq)$ if it is totally ordered by $\leq$: $(x \leq y) \vee (y \leq x)\ \forall x, y \in C$.

Choose non-empty sets T, X, and Y, and fix non-empty sets $\mathcal{T}$, Ω, and Z, such that $\mathcal{T} \in \mathcal{P}'(\mathcal{P}'(T))$, $\Omega \in \mathcal{P}'(Y^T)$, and $Z \in \mathcal{P}'(X^T)$.

We call a multi-function $\alpha \in \mathcal{P}(Z)^\Omega$ *non-anticipating* if the relation

$$((\omega_1\,|\,H) = (\omega_2\,|\,H)) \Rightarrow \big(\big(\alpha(\omega_1)\,|\,H\big) = \big(\alpha(\omega_2)\,|\,H\big)\big) \quad (1)$$

is fulfilled for all $\omega_1, \omega_2 \in \Omega$, $H \in \mathcal{T}$ and denote by $\mathbf{N}$ the set of all non-anticipating multi-functions from Ω to Z.

We call a function $\beta \in Z^\Omega$ non-anticipating if the relation

$$((\omega_1\,|\,H) = (\omega_2\,|\,H)) \Rightarrow ((\beta(\omega_1)\,|\,H) = (\beta(\omega_2)\,|\,H)) \quad (2)$$

is fulfilled for all $\omega_1, \omega_2 \in \Omega$, $H \in \mathcal{T}$ and denote by $\mathbf{n}$ the set of all non-anticipating functions from Ω to Z.

For any $\alpha \in \mathcal{P}(Z)^\Omega$, we define the subset $\mathbf{n}[\alpha] \in \mathcal{P}(\mathbf{n})$ of functions that are also selections of α:

$$\mathbf{n}[\alpha] \triangleq \mathbf{n} \cap \prod_{\omega \in \Omega} \alpha(\omega). \quad (3)$$

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Thus, $\mathbf{n}[\alpha] \in \mathcal{P}(Z^{\Omega})$ is the set of all non-anticipating selections of a multi-function α.

Let the set X be equipped with the Hausdorff topology τ_X. Then we assume that the set $Z \in \mathcal{P}'(X^T)$ is equipped with the topology τ_Z induced by the Tikhonov topology $\otimes^T(\tau_X)$ on the product $\prod_{t\in T} X_t$, $X_t \triangleq X$; we also assume that the set $Z^{\Omega} = \prod_{\omega\in\Omega} Z_\omega$, $Z_\omega \triangleq Z$ is equipped by the Tikhonov product topology $\tau_{Z^\Omega} \triangleq \otimes^{\Omega}(\tau_Z)$.

3. THEOREM AND EXAMPLES

In Serkov and Chentsov (2018) the following theorem is proved.

Theorem 1. Let $\mathcal{T}$ be a chain in the poset $(\mathcal{P}(T), \subset)$. Let $\alpha \in \mathbf{N}$ and $\alpha(\omega)$ be non-empty compact in (Z, τ_Z) for every $\omega \in \Omega$. Then $\mathbf{n}[\alpha]$ is a non-empty compact in $(Z^{\Omega}, \tau_{Z^\Omega})$.

In particular, every non-anticipating multi-function with non-empty compact values has a non-anticipating selection.

In the following examples, we choose the domain of "time" variable as a segment of the real line that is typical for control problems. The first example shows that the total ordering condition on the family $\mathcal{T}$ in Theorem 1 is essential. The second one shows that replacing this condition with the condition «$\mathcal{T}$ forms a base of a filter» does not improve the situation.

Example 1. Let $T \triangleq [-\pi, \pi]$, $X = Y \triangleq \mathbb{R}$ and the set X be equipped with the topology τ_X, generated by $|\cdot|$-metric. So, (X, τ_X) is T_2-topological space. Let $A \triangleq [-\pi, -\frac{3\pi}{4}]$, $B \triangleq [\frac{3\pi}{4}, \pi]$ and $\mathcal{T} \triangleq \{A, B\}$. Then, $\mathcal{T} \in \mathcal{P}'(\mathcal{P}'(T))$. Let elements $\omega_r, \omega_b, \omega_g, \omega_y \in Y^T$ be defined by the relations (see Fig. 1):

$$\begin{gathered} \omega_r(t) \triangleq \max\{\omega_b(t), \omega_y(t), \pi/2 - \cos(t)\}, \\ \omega_b(t) \triangleq t,\ \omega_g(t) \triangleq -\omega_r(t),\ \omega_y(t) \triangleq -\omega_b(t) \end{gathered} \tag{4}$$

for all $t \in T$ and let $\Omega \triangleq \{\omega_r, \omega_b, \omega_g, \omega_y\}$. It is easy to

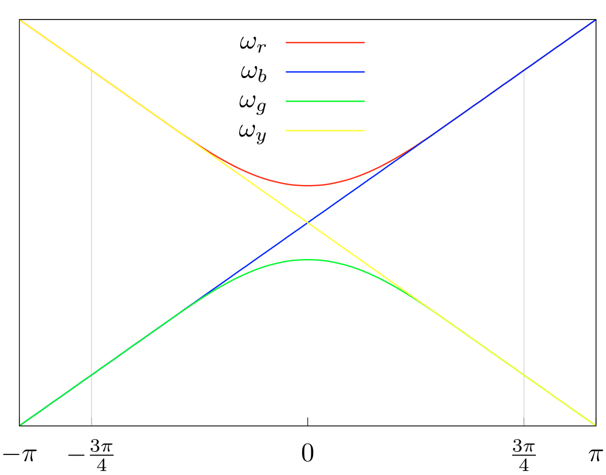

Fig. 1. The set Ω: the case $A \cap B = \varnothing$.

verify that the restrictions of elements (4) to the sets A and B satisfy only the following relations:

$$(\omega_r \,|\, B) = (\omega_b \,|\, B), \tag{5}$$
$$(\omega_b \,|\, A) = (\omega_g \,|\, A), \tag{6}$$
$$(\omega_g \,|\, B) = (\omega_y \,|\, B), \tag{7}$$
$$(\omega_y \,|\, A) = (\omega_r \,|\, A). \tag{8}$$

Let elements $h_{r1}, h_{b1}, h_{g1}, h_{y1}, h_{r2}, h_{b2}, h_{g2}, h_{y2} \in X^T$ be defined by the relations (see Fig. 2; in the definitions of h_{r1}, h_{r2} we use extensions of ω_r provided by (4)):

$$\begin{aligned} h_{r1}(t) &\triangleq \omega_r(t + \pi/4) + \pi/4, & h_{r2}(t) &\triangleq \omega_r(t - \pi/4) + \pi/4, \\ h_{b1}(t) &\triangleq \omega_b(t), & h_{b2}(t) &\triangleq \omega_b(t) + \pi/2, \\ h_{g1}(t) &\triangleq \omega_g(t), & h_{g2}(t) &\triangleq \omega_g(t) + \pi/2, \\ h_{y1}(t) &\triangleq \omega_y(t), & h_{y2}(t) &\triangleq \omega_y(t) + \pi/2, \end{aligned}$$

for $t \in T$ and let $Z \triangleq \{h_{r1}, h_{b1}, h_{g1}, h_{y1}, h_{r2}, h_{b2}, h_{g2}, h_{y2}\}$. It is easy to verify that the restrictions of the elements from Z to the sets A and B satisfy the relations below:

$$(h_{r1} \,|\, B) = (h_{b2} \,|\, B), \quad (h_{r2} \,|\, B) = (h_{b1} \,|\, B), \tag{9}$$
$$(h_{b1} \,|\, A) = (h_{g1} \,|\, A), \quad (h_{b2} \,|\, A) = (h_{g2} \,|\, A), \tag{10}$$
$$(h_{g1} \,|\, B) = (h_{y1} \,|\, B), \quad (h_{g2} \,|\, B) = (h_{y2} \,|\, B), \tag{11}$$
$$(h_{y1} \,|\, A) = (h_{r1} \,|\, A), \quad (h_{y2} \,|\, A) = (h_{r2} \,|\, A). \tag{12}$$

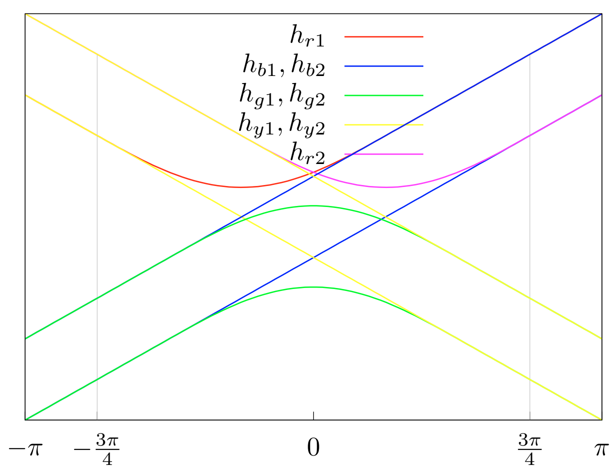

Fig. 2. The set Z: the case $A \cap B = \varnothing$.

We consider the multi-function $\alpha \in \mathcal{P}'(Z)^{\Omega}$ of the form

$$\alpha(\omega_r) \triangleq \{h_{r1}, h_{r2}\}, \tag{13}$$
$$\alpha(\omega_b) \triangleq \{h_{b1}, h_{b2}\}, \tag{14}$$
$$\alpha(\omega_g) \triangleq \{h_{g1}, h_{g2}\}, \tag{15}$$
$$\alpha(\omega_y) \triangleq \{h_{y1}, h_{y2}\}. \tag{16}$$

It is clear that the values of the multi-function are non-empty and compact in (Z, τ_Z) (as, in fact, in any other topology on Z). From relations (9)–(12) and (13)–(16), we derive the following equalities:

$$\big(\alpha(\omega_r) \,|\, B\big) = \big(\alpha(\omega_b) \,|\, B\big), \tag{17}$$
$$\big(\alpha(\omega_b) \,|\, A\big) = \big(\alpha(\omega_g) \,|\, A\big), \tag{18}$$
$$\big(\alpha(\omega_g) \,|\, B\big) = \big(\alpha(\omega_y) \,|\, B\big), \tag{19}$$
$$\big(\alpha(\omega_y) \,|\, A\big) = \big(\alpha(\omega_r) \,|\, A\big). \tag{20}$$

It means (see (5)–(8), (1)) that α is non-anticipative, or $\alpha \in \mathbf{N}$. Thus, α satisfies all the conditions of Theorem 1.

Let us show that $\mathbf{n}[\alpha] = \varnothing$, i.e. there is no non-anticipating selections of the multi-function α. Suppose the contrary, that there exists a function $\beta \in \mathbf{n}[\alpha]$. Hence (see (3), (13)–(16)), β satisfies (2) and the inclusions

$$\beta(\omega_r) \in \{h_{r1}, h_{r2}\}, \tag{21}$$
$$\beta(\omega_b) \in \{h_{b1}, h_{b2}\}, \tag{22}$$
$$\beta(\omega_g) \in \{h_{g1}, h_{g2}\}, \tag{23}$$
$$\beta(\omega_y) \in \{h_{y1}, h_{y2}\}. \tag{24}$$

Let us suppose (see (21)) that

$$\beta(\omega_r) = h_{r1}. \tag{25}$$

From (2), (5), (25), we have $(h_{r1}\,|\,B) = (\beta(\omega_r)\,|\,B) = (\beta(\omega_b)\,|\,B)$. Then (see (22), (9)),

$$\beta(\omega_b) = h_{b2}. \tag{26}$$

From (2), (6), (26), we have $(h_{b2}\,|\,A) = (\beta(\omega_b)\,|\,A) = (\beta(\omega_g)\,|\,A)$. Then (see (23), (10)),

$$\beta(\omega_g) = h_{g2}. \tag{27}$$

From (2), (7), (27), we have $(h_{g2}\,|\,B) = (\beta(\omega_g)\,|\,B) = (\beta(\omega_y)\,|\,B)$. Then (see (24), (11)),

$$\beta(\omega_y) = h_{y2}. \tag{28}$$

From (2), (8), (28), we have $(h_{y2}\,|\,A) = (\beta(\omega_y)\,|\,A) = (\beta(\omega_r)\,|\,A)$. Then (see (21), (12)),

$$\beta(\omega_r) = h_{r2}, \tag{29}$$

which contradicts equality (25). So, assuming $\beta(\omega_r) = h_{r1}$, we get a contradiction. In view of (21), there is only one possibility: to assume that $\beta(\omega_r) = h_{r2}$. But in this case, by analogous arguments, we obtain the equality $\beta(\omega_r) = h_{r1}$, which contradicts this assumption. Thus, the assumption of the existence of a non-anticipative selection of the multi-function α leads to contradiction, that is, the statement of Theorem 1 does not hold.

Example 2. A similar example holds in the case when the family $\mathcal{T}$ forms a base of a filter. Recall (see, for example, (Engelking, 1985, §1.6.7)) that a family $\mathcal{V} \in \mathcal{P}'(\mathcal{P}'(V))$ is a filter base if, for any $v, v' \in \mathcal{V}$, there exists a set $v'' \in \mathcal{V}$ such that $v'' \subset v \cap v'$.

Let T, X, Y and τ_X be defined as in Example 1. Let $A \triangleq [-\pi, \pi/3]$, $B \triangleq [-\pi/3, \pi]$, $C \triangleq [-\pi/3, \pi/3]$ and $\mathcal{T} \triangleq \{A, B, C\}$. Then $\mathcal{T} \in \mathcal{P}'(\mathcal{P}'(T))$, and the family $\mathcal{T}$ is a filter base. Let the elements $\omega_r, \omega_b, \omega_g, \omega_y \in Y^T$ be of the form (see Fig. 3; for clarity, these functions are shown slightly apart from each other):

$$\omega_b(t) \triangleq \max\{\min\{t + \pi/3, 0\}, t - \pi/3\}, \quad \omega_y(t) \triangleq -\omega_b(t),$$
$$\omega_r(t) \triangleq \max\{-t - \pi/3, 0, t - \pi/3\}, \qquad \omega_g(t) \triangleq -\omega_r(t),$$

$t \in T$, and let $\Omega \triangleq \{\omega_r, \omega_b, \omega_g, \omega_y\}$. It is easy to verify that the restrictions of the elements from Ω to the set C coincide:

$$(\omega_r\,|\,C) = (\omega_b\,|\,C) = (\omega_g\,|\,C) = (\omega_y\,|\,C), \tag{30}$$

and the restrictions to the sets A and B (as in Example 1) satisfy only relations (5)–(8).

Let the elements $h_{r1}, h_{b1}, h_{g1}, h_{y1}, h_{r2}, h_{b2}, h_{g2}, h_{y2} \in X^T$ be of the form (see Fig. 4):

$$\begin{aligned}
&h_{r1}(t) \triangleq \max\{-0.7 \times (t + \pi/3), 0, t - \pi/3\},\\
&h_{r2}(t) \triangleq \max\{-t - \pi/3, 0, 0.7 \times (t - \pi/3)\},\\
&h_{b1}(t) \triangleq \max\{\min\{t + \pi/3, 0\}, 0.7 \times (t - \pi/3)\},\\
&h_{b2}(t) \triangleq -h_{y1}(t),\\
&h_{g1}(t) \triangleq \omega_g(t),\\
&h_{g2}(t) \triangleq 0.7 \times \omega_g(t),\\
&h_{y1}(t) \triangleq \max\{-0.7 \times (t + \pi/3), \min\{0, \pi/3 - t\}\},\\
&h_{y2}(t) \triangleq -h_{b1}(t),
\end{aligned} \tag{31}$$

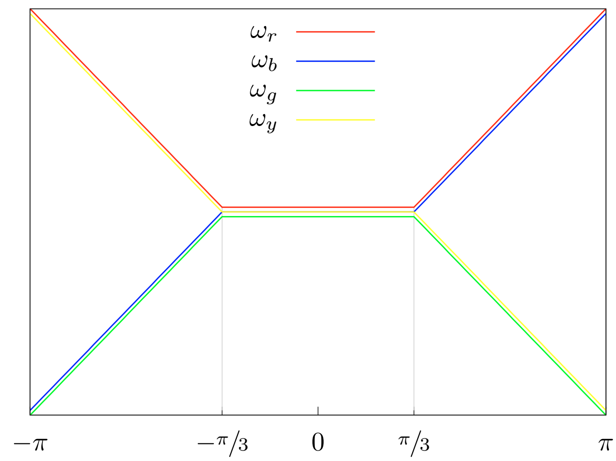

Fig. 3. The set Ω: $\mathcal{T}$ is a base of a filter.

$t \in T$, and let $Z \triangleq \{h_{r1}, h_{b1}, h_{g1}, h_{y1}, h_{r2}, h_{b2}, h_{g2}, h_{y2}\}$. Then the restrictions of elements (31) to the set C coincide:

$$\begin{aligned}(h_{r1}\,|\,C) = (h_{r2}\,|\,C) = (h_{b1}\,|\,C) = (h_{b2}\,|\,C) =\\ = (h_{g1}\,|\,C) = (h_{g2}\,|\,C) = (h_{y1}\,|\,C) = (h_{y2}\,|\,C),\end{aligned} \tag{32}$$

and the restrictions to the sets A and B (as in Example 1) satisfy relations (9)–(12). Let a multi-function $\alpha \in \mathcal{P}'(Z)^{\Omega}$

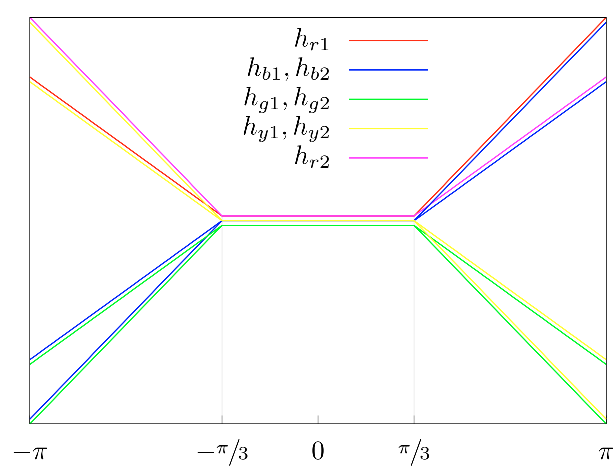

Fig. 4. The set Z: $\mathcal{T}$ is a base of a filter.

be defined by relations (13)–(16). It is clear, that the values of α are non-empty and compact in (Z, τ_Z). From (9)–(12), (32) and (13)–(16), we have equalities (17)–(20) and

$$\big(\alpha(\omega_r)\,|\,C\big) = \big(\alpha(\omega_b)\,|\,C\big) = \big(\alpha(\omega_g)\,|\,C\big) = \big(\alpha(\omega_y)\,|\,C\big). \tag{33}$$

From relations (5)–(8), (30), (17)–(20), (33) and definition (1), it follows that the multi-function α is non-anticipating, or $\alpha \in \mathbf{N}$. We have verified that the multi-function α satisfies all the conditions of Theorem 1.

To show that $\mathbf{n}[\alpha] = \varnothing$, let us suppose the contrary: there is a function $\beta \in \mathbf{n}[\alpha]$. Then, as in Example 1, assumption (25) implies equality (29), contradicting this assumption, and vice versa. So, the assumption $\beta \in \mathbf{n}[\alpha]$ was wrong.

Thus, the weakening of the total ordering condition to the filter base condition in Theorem 1 is also impossible.

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